

A Generalization of Brown's Construction for the Degree/Diameter Problem

Yawara ISHIDA

December 31, 2015

Abstract

The degree/diameter problem is the problem of finding the largest possible number of vertices $n_{\Delta,D}$ in a graph of given degree Δ and diameter D . We consider the problem for the case of diameter $D = 2$. William G Brown gave a lower bound of the order of $(\Delta, 2)$ -graph. In this paper, we give a generalization of his construction and improve the lower bounds for the case of $\Delta = 306$ and $\Delta = 307$. One is $(306, 2)$ -graph with 88723 vertices, the other is $(307, 2)$ -graph with 88724 vertices.

1 Introduction

The *degree/diameter problem* is the problem of finding the largest possible number $n_{\Delta,D}$ of vertices in a graph of given degree Δ and diameter D [3, 1]. Let G be a graph with degree Δ ($\Delta > 2$) and diameter D , then we have

$$|G| \leq n_{\Delta,D} \leq 1 + \Delta \sum_{k=1}^{D-1} (\Delta - 1)^k$$

where $|G|$ is the number of vertices of G . The right hand side of the above inequation is called *Moore bound*. On the other hand, a lower bound of $n_{\Delta,D}$ for small degree and small diameter are available at <http://combinatoricswiki.org>. Especially for case of $D = 2$ and large degree, there exists the general construction that gives a lower bound of $n_{\Delta,2}$, which is called *Brown's construction* [3]. Let F_q be the finite field, where q is a power of a prime. Brown's construction gives the graph $B(F_q)$ whose vertices are lines in F_q^3 and two lines are adjacent if and only if they are orthogonal. It follows that

$$|B(F_q)| = q^2 + q + 1, \quad \Delta(B(F_q)) = q + 1, \quad D(B(F_q)) = 2.$$

The degree of each vertex of $B(F_q)$ is $q + 1$ or q . Among $q^2 + q + 1$ vertices, $q + 1$ vertices are of degree q and q^2 vertices are of degree $q + 1$. If q is a power of 2, there exists $(q + 1, 2)$ -graph with $q^2 + q + 2$ vertices [2].

In this paper, we generalize Brown's construction by replacing a field with a commutative ring, and search new records of the degree/diameter problem.

2 Generalized Brown's Construction

We give some definitions for generalized Brown's construction. A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set $E \subset \{(v, w) \in V^2 | v \neq w\}$ of *edges*. If (v, w) is in E , it is said that v and w are *adjacent*, which is denoted by $v \sim w$. The *order* $|G|$ of the graph is the number of vertices. The *neighbours* $N(v)$ of the vertex v is a set of vertices which are adjacent to v . The *degree* $\delta(v)$ of the vertex v is the number of neighbours $|N(v)|$. The *degree* $\Delta(G)$ of the graph G is the maximum degree of vertices, namely $\Delta(G) = \max\{\delta(v) | v \in V\}$. The *distance* of the pair (v, w) of vertices is the shortest path length between v and w . The *diameter* $D(G)$ of the graph is the maximum distance of all pairs of vertices.

Let R be a commutative ring with unity. R^* denotes the set of invertible elements of R . R^3 is naturally seen as R -module. The addition and R -action are defined by coordinate-wise. The *inner product* $\cdot : R^3 \times R^3 \Rightarrow R$ is defined as follows:

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

\mathbf{v} and \mathbf{w} are *orthogonal* if and only if the inner product vanishes, namely $\mathbf{v} \cdot \mathbf{w} = 0$. The *cross product* $\times : R^3 \times R^3 \Rightarrow R$ is defined as follows:

$$(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Definition 1. Let R be a commutative ring with unity. The vertex set V of the graph $B(R)$ is

$$V = (R^3 \setminus \{\mathbf{v} | \exists r \in R, r \cdot \mathbf{v} = \mathbf{0}\}) / \sim$$

where $\mathbf{v} \sim \mathbf{w}$ if and only if there exists $k \in R^*$ such that $k \cdot \mathbf{v} = \mathbf{w}$. The two vertices $[\mathbf{v}]$ and $[\mathbf{w}]$ are *adjacent* if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

The definition of adjacency above is well-defined because the orthogonality does not depend on the selection of representatives. We call the above construction of a graph from a ring *generalized Brown's construction*. It is clear that the new construction coincides with Brown's one when the ring R is a field.

Lemma 1. Let E be a Euclidean domain and u be a prime element in E . If $E/(u^k)$ is a finite ring, then the degree of each vertex of $B(E/(u^k))$ is Δ or $\Delta - 1$, where (u^k) is the principal ideal generated by u^k .

Proof. It is clear that the degrees of vertices represented by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are the same. Let $\mathbf{v} = ([a], [b], [c])$ be a representative of any vertex where $a, b, c \in E$. If any element of $[a], [b], [c]$ is not invertible in $E/(u^k)$, there exist natural numbers $1 \leq l, m, n < k$ and some elements a', b', c' in E such that $a = u^l a', b = u^m b', c = u^n c'$. \mathbf{v} is not a representative of vertices because the equation $[u^{\min(l, m, n)}] \cdot \mathbf{v} = ([0], [0], [0])$ holds. This is a contradiction. Therefore, at least one of $[a], [b], [c]$ is invertible. If $[a]$ is invertible, there exists one-to-one correspondence $\bar{U} : N([(1, 0, 0)]) \rightarrow N([\mathbf{v}])$ such that for all $[\mathbf{w}] \in N([(1, 0, 0)])$,

$\overline{U}([w]) = [U^{-1}w]$ where U is an invertible matrix defined as follows

$$U = \begin{pmatrix} [a] & 0 & 0 \\ [b] & 1 & 0 \\ [c] & 0 & 1 \end{pmatrix}$$

If $v \cdot v = 0$, then $\delta([v]) = \delta([(1, 0, 0)]) - 1$. If not so, $\delta([v]) = \delta([(1, 0, 0)])$. In the same way, if $[b]/[c]$ is invertible, then $\delta([v])$ is $\delta([(0, 1, 0)])/\delta([(0, 0, 1)])$ or $\delta([(1, 0, 0)])/\delta([(0, 0, 1)]) - 1$. Therefore, for all the vertex $[v]$,

$$\delta([v]) = \begin{cases} \delta([(1, 0, 0)]) & (v \cdot v \neq 0) \\ \delta([(1, 0, 0)]) - 1 & (otherwise) \end{cases}$$

□

Lemma 2. *Let E be a Euclidean domain and I be an ideal of E . The diameter of $B(E/I)$ is 2.*

Proof. For any two distinct vertices represented by $v = ([v_1], [v_2], [v_3])$ and $w = ([w_1], [w_2], [w_3])$, consider the cross product $v \times w$. If $v \times w = \mathbf{0}$, then $[v_i] \cdot w = [w_i] \cdot v$ for $i = 1, 2, 3$. There exists $e \in E$ such that $I = (e)$ because any Euclidean domain is a principal ideal domain. If $\gcd(v_1, v_2, v_3, e)$ is not a unity, where \gcd is a greatest common divisor, there exists $e' \neq 1$ in E such that $e = de'$. v is not a representative because $[e'] \cdot v = \mathbf{0}$. This is a contradiction. Therefore d is a unity, namely v_1 and v_2, v_3, e are coprime. Then there exist $a, b, c, d \in E$ such that $av_1 + bv_2 + cv_3 + de = 1$ in E . Seeing this formula in E/I , we get $[a][v_1] + [b][v_2] + [c][v_3] = [1]$.

$$v = [1] \cdot v = ([a][v_1] + [b][v_2] + [c][v_3])v = ([a][w_1] + [b][w_2] + [c][w_3])w$$

means $[v] = [w]$, which is a contradiction to that two vertices are distinct, then $v \times w \neq \mathbf{0}$. If $v \times w$ is a representative of vertex, $[v \times w]$ is adjacent to $[v]$ and $[w]$. If $v \times w = ([k_1], [k_2], [k_3])$ is not a representative of vertex, $v \times w = [\gcd(k_1, k_2, k_3)] \cdot u$ and u is a representative of vertex. $[u]$ is adjacent to $[v]$ and $[w]$. □

Theorem. *The following equations hold.*

1. $|B(\mathbb{Z}_{p^k})| = p^{2k} + p^{2k-1} + p^{2k-2}$
2. $\Delta(B(\mathbb{Z}_{p^k})) = p^k + p^{k-1}$
3. $D(\mathbb{Z}_{p^k}) = 2$

Proof. It is straightforward to show the formula of the order of $B(\mathbb{Z}_{p^k})$.

$$\begin{aligned} |B(\mathbb{Z}_{p^k})| &= \frac{|\mathbb{Z}_{p^k}|^3 - |\{mp | 0 \leq m < p^{k-1}\}|^3}{|\mathbb{Z}_{p^k}| - |\{mp | 0 \leq m < p^{k-1}\}|} \\ &= \frac{(p^k)^3 - (p^{k-1})^3}{p^k - p^{k-1}} = p^{2k} + p^{2k-1} + p^{2k-2} \end{aligned}$$

Using Lemma 1, it is only enough to show that the degree of the vertex represented by $(1, 0, 0)$ satisfy the formula of the degree of $B(\mathbb{Z}_{p^k})$.

$$\begin{aligned}\Delta(B(\mathbb{Z}_p^k)) &= \delta([(1, 0, 0)]) = \frac{|\mathbb{Z}_{p^k}|^2 - |\{mp | 0 \leq m < k\}|^2}{|\mathbb{Z}_{p^k}| - |\{mp | 0 \leq m < k\}|} \\ &= \frac{(p^k)^2 - (p^{k-1})^2}{p^k - p^{k-1}} = p^k + p^{k-1}\end{aligned}$$

Using Lemma 2, we get $D(B(\mathbb{Z}_p^k)) = 2$

□

We search new records of the degree/diameter problem among graphs by generalized Brown's construction of \mathbb{Z}_n where $2 \leq n \leq 10000$. Using the above theorem, $B(\mathbb{Z}_{17^2})$ has degree 306 and diameter 2 and 88723 vertices. It is a new record of $(306, 2)$ of the degree/diameter problem because it cannot be obtained from ordinary Brown's construction. The power of a prime less than $305 = 306 - 1$ is 293^1 and the graph $B(\mathbb{Z}_{293})$ obtained from ordinary Brown's construction of 293^1 has $294 = 293 + 1$ degree and $86143 = 293^2 + 293 + 1$ vertices. The old record of $306 = 294 + 12$ is $86156 = 86143 + 12$ obtained from $B(\mathbb{Z}_{293})$ by duplicating vertices. In the same way, the graph obtained from $B(\mathbb{Z}_{17^2})$ by duplicating any one vertex, whose order is 88724, is a new record of $(307, 2)$ because the power of a prime less than $306 = 307 - 1$ is 293^1 .

Acknowledgements

We wish to thank Ryosuke Mizuno, Nobuhito Tamaki, Masahito Hasegawa, Shinya Katsumata, and Sakie Suzuki.

References

- [1] William G Brown. On graphs that do not contain a thomsen graph. *Canad. Math. Bull.*, 9(2):1–2, 1966.
- [2] Paul Erdős, Siemion Fajtlowicz, and Alan J. Hoffman. Maximum degree in graphs of diameter 2. *Networks*, 10(1):87–90, 1980.
- [3] M. Miller and J. Širáň. Moore graphs and beyond: a survey of the degree/diameter problem. *The Electronic Journal of Combinatorics*, (DS14), 2005.